# A GENERAL ALGORITHM FOR RATIONAL INTERPOLATION

by

S. L. Loi

No. 33 June, 1984

Abstract A recursive algorithm for the construction of the generalized form of the interpolating rational function is derived. This generalization of the Neville-Aitken algorithm constructs a table of all possible rational interpolants in implicit form. The algorithm may be simply modified so that it does not break down when a singularity occasionally appears. The coefficients of the interpolant and the evaluation of the interpolant at an arbitrary point may be easily calculated.

A GENERAL ALGORITHM FOR RATIONAL INTERPOLATION

## 1 INTRODUCTION

Recently Brezinski [1] presented a very effective general extrapolation algorithm for linear and rational extrapolation. This algorithm was extended by Brezinski to a general interpolation algorithm which is called the Muhlbach-Neville-Aitken (MNA) Algorithm in [2]. In this report, the rational interpolating function is generalized in a similar way. This method answers the question raised in [2] by Brezinski. The aim of this paper is to present an algorithm for recursively constructing the interpolating rational function.

$$R^{i}_{m,n}(x) = \frac{\sum_{j=0}^{m} a_j\, h_j(x)}{\sum_{j=0}^{n} b_j\, h_j(x)} \qquad (1.1)$$

With $R^{i}_{m,n}(x_j) = f_j$, $j = i, i+1 \ldots i+m+n$, $i=0,1,2\ldots$, from either $R^{i}_{m,n-1}(x)$ or $R^{i}_{m-1,n}(x)$ by an extension of the MNA algorithm. The set of given functions $[\{h_j(x)\}_{j=0}^{m}, \{h_j(x) f(x)\}_{j=1}^{n}]$ is assumed to form a complete or quasicomplete Chebyshev system [7] on the set of interpolating points.

Instead of setting $R^{i}_{n,0}(x) = \dfrac{1}{R^{i}_{0,n}(x)}$ $\quad n = 1,2 \ldots$

for the rational function (as in [2]), we apply this algorithm to the rational fraction $R^{i}_{0,n}(x)$ directly. Then the $R^{i}_{m,n}(x)$ can be generated either from the (m-1)th row vertically or from the (n-1)th column horizontally. If (N+1) interpolating points are given, we could construct the following triangular array of the interpolating function

$$\begin{array}{llll}
R^i_{0,0} & R^i_{0,1} \cdots & R^i_{0,n-1} & R^i_{0,n} \\
R^i_{1,0} & R^i_{1,1} \cdots & R^i_{1,n-1} & \\
\cdot & \cdot & & \\
\cdot & \cdot & & \\
\cdot & \cdot & & \\
R^i_{n-1,0} & R^i_{n-1,1} & & \\
R^i_{n,0} & & &
\end{array}$$

Figure 1,1

In this array, each term $R^i_{m,n}$ represents a table of functions for $i = 0,1\ldots, N-m-n$.

This algorithm is more general than the algorithms suggested by Larkin [6]. With the normalization $b_0 = 1$, $R^i_{m,n}(x)$ is expressed implicitly in the form

$$h_0(x)R^i_{m,n}(x) = R^i_{m,n}(x)\sum_{j=1}^{n} b_j h_j(x) + \sum_{j=0}^{m} a_j h_j(x) \tag{1.2}$$

By using this algorithm, the coefficients of the interpolating function can be determined easily. The interpolating value at a given point x=a can be found implicitly. However for computational convenience, we could transform the representation to a new basis $\bar{h}_j(x) = h_j(x) - h_j(a)$, $\forall j > 0$, so that $\bar{h}_j(a) = 0$, $\forall j > 0$. In this case interpolation will be reduced to general rational extrapolation.

In Section 2 the MNA algorithm [2] is reviewed. The extension of this algorithm to generalized rational functions is given in Section 3. In Section 4 we discuss how to overcome the problem when the algorithm breaks down due to a zero divisor. Some examples and numerical results are given in Section 5.

## 2 THE MNA ALGORITHM

We begin by reviewing the MNA algorithm [2].

Suppose we construct the polynomials

$$P_k^n(x) = a_0 g_0(x) + \dots + a_k g_k(x) \quad ,$$

$$\text{such that } P_k^n(x_i) = f_i \;,$$

where $(x_i, f_i)$ $i = n, \dots, n+k$ are the interpolating points.

Then $P_k^n(x)$ can be expressed as

$$P_k^n(x) = \frac{\begin{vmatrix} 0 & -f_n & \cdots & -f_{n+k} \\ g_0(x) & g_0(x_n) & \cdots & g_0(x_{n+k}) \\ \cdots & \cdots & \cdots & \cdots \\ g_k(x) & g_k(x_n) & \cdots & g_k(x_{n+k}) \end{vmatrix}}{D} \qquad (2.1)$$

Note: In this paper D is the minor obtained by eliminating the first row and the first column of the determinant.

Now let $g_{k,i}^n(x)$ be the ratio of determinants obtained replacing the first row in the numerator of $P_k^n$ by

$$(-g_i(x) \quad -g_i(x_n) \quad \dots \quad -g_i(x_{n+k}))$$

Note that $g^n_{k,i}(x) = 0,\ i \leqslant k$ and $g^n_{k,i}(x_j) = 0,\ j = n,\dots,n+k$.

Then the MNA algorithm is the following :

$$P^n_0(x) = \frac{\begin{vmatrix} 0 & -f_n \\ g_0(x) & g_0(x_n) \end{vmatrix}}{g_0(x_n)} = f_n \frac{g_0(x)}{g_0(x_n)} ,$$

$$g^n_{0,i}(x) = \frac{\begin{vmatrix} -g_i(x) & -g_i(x_n) \\ g_0(x) & g_0(x_n) \end{vmatrix}}{g_0(x_n)} = g_i(x_n)\frac{g_0(x)}{g_0(x_n)} - g_i(x), \quad i = 1,2 \dots$$

For $k = 1,2 \dots$ and $n = 0,1 \dots$

$$P^n_k(x) = \frac{g^{n+1}_{k-1,k}(x)\ P^n_{k-1}(x) - g^n_{k-1,k}(x)\ P^{n+1}_{k-1}(x)}{g^{n+1}_{k-1,k}(x) - g^n_{k-1,k}(x)}$$

$$= P^n_{k-1}(x) - \frac{\Delta P^n_{k-1}(x)}{\Delta g^n_{k-1}(x)}\, g^n_{k-1,k}(x) \qquad (2.2a)$$

$$g^n_{k,i}(x) = \frac{g^{n+1}_{k-1,k}(x)\ g^n_{k-1,i}(x) - g^n_{k-1,k}(x)\ g^{n+1}_{k-1,i}(x)}{g^{n+1}_{k-1,k}(x) - g^n_{k-1,k}(x)}$$

$$= g^n_{k-1,i}(x) - \frac{\Delta g^n_{k-1,i}(x)}{\Delta g^n_{k-1,k}(x)}\, g^n_{k-1,k}(x) \quad i = k+1,k+2 \dots \qquad (2.2b)$$

and $\Delta$ is the forward difference on the index n.
The way of computing $g^n_{k,i}(x)$ in [2] is equivalent to Aitken's pattern. With the initialization of $g^n_{0,i}(x)$ for a fixed value of n, it can be shown in Figure 2.1.

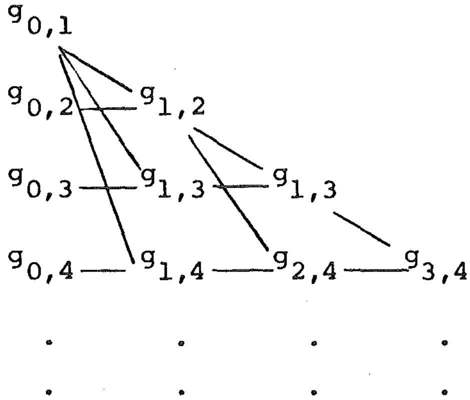



Figure 2.1

To show the relationship to its generalization, this algorithm is essentially restated in a new notation.

For convenience, instead of using $g^n_{k,i}(x)$, we initialize

$$Vg^n_{1-k,k}(x) = Hg^n_{1-k,k}(x) = \frac{\begin{vmatrix} -g_k(x) & -g_k(x_n) \\ g_0(x) & g_0(x_n) \end{vmatrix}}{g_0(x_n)} \qquad (2.3a)$$

$$k = 1,2 \dots.$$

Note: $Vg^n_{k,0}(x)$ is used to generate $P^n_k(x)$ vertically.

$Hg^n_{0,k}(x)$ is used to generate $P^n_k(x)$ horizontally.

We define $Hg^n_{-j,k}(x) = 0$ and $Vg^n_{-j,k}(x) = 0$ when $j \geqslant k$.

For $i = 2,3,4 \dots$, $j = 0,1,2 \dots$, and $k = j+i$.

$$Hg^n_{-j,k}(x) = \frac{Hg^n_{-(j+1),k}(x)\, Vg^{n+1}_{-j,k-1}(x) - Hg^{n+1}_{-(j+1),k}(x)\, Vg^n_{-j,k-1}(x)}{Vg^{n+1}_{-j,k-1}(x) - Vg^n_{-j,k-1}(x)}$$

$$= Hg^n_{-(j+1),k}(x) - \frac{\Delta\, Hg^n_{-(j+1),k}(x)}{\Delta\, Vg^n_{-j,k-1}(x)}\, Vg^n_{-j,k-1}(x) \qquad (2.3b)$$

$$Vg^{n}_{-j,k}(x) = \frac{Vg^{n}_{-j,k-1}(x)\ Hg^{n+1}_{-(j+1),k}(x) - Vg^{n+1}_{-j,k-1}(x)\ Hg^{n}_{-(j+1),k}(x)}{Hg^{n+1}_{-(j+1),k}(x) - Hg^{n}_{-(j+1),k}(x)}$$

$$= Vg^{n}_{-j,k-1}(x) - \frac{\Delta\ Vg^{n}_{-j,k-1}(x)}{\Delta\ Hg^{n}_{-(j+1),k}(x)}\ Hg^{n}_{-(j+1),k}(x) \qquad (2.3c)$$

where
$$Hg^{n}_{-j,k}(x) = \frac{\begin{vmatrix} -g_k(x) & -g_k(x_n) & \cdots & -g_k(x_{n+k-j-1}) \\ g_0(x) & g_0(x_n) & \cdots & g_0(x_{n+k-j-1}) \\ g_{j+1}(x) & g_{j+1}(x_n) & \cdots & g_{j+1}(x_{n+k-j-1}) \\ \cdots & \cdots & \cdots & \cdots \\ g_{k-1}(x) & g_{k-1}(x_n) & \cdots & g_{k-1}(x_{n+k-j-1}) \end{vmatrix}}{\begin{vmatrix} g_0(x_n) & \cdots & g_0(x_{n+k-j-1}) \\ g_{j+1}(x_n) & \cdots & g_{j+1}(x_{n+k-j-1}) \\ \cdots & \cdots & \cdots \\ g_{k-1}(x_n) & \cdots & g_{k-1}(x_{n+k-j-1}) \end{vmatrix}} \qquad (2.3d)$$

$$Vg^{n}_{-j,k}(x) = \frac{\begin{vmatrix} -g_k(x) & -g_k(x_n) & \cdots & -g_k(x_{n+k-j-1}) \\ g_0(x) & g_0(x_n) & \cdots & g_0(x_{n+k-j-1}) \\ g_{j+1}(x) & g_{j+1}(x_n) & \cdots & g_{j+1}(x_{n+k-j-1}) \\ \cdots & \cdots & \cdots & \cdots \\ g_{k-1}(x) & g_{k-1}(x_n) & \cdots & g_{k-1}(x_{n+k-j-1}) \end{vmatrix}}{\begin{vmatrix} g_0(x_n) & \cdots & g_0(x_{n+k-j-1}) \\ g_{j+2}(x_n) & \cdots & g_{j+2}(x_{n+k-j-1}) \\ \cdots & \cdots & \cdots \\ g_k(x_n) & \cdots & g_k(x_{n+k-j-1}) \end{vmatrix}} \qquad (2.3e)$$

and
$$Hg^{n}_{0,k}(x) = \frac{\begin{vmatrix} -g_k(x) & -g_k(x_n) & \cdots & -g_0(x_{n+k-j-1}) \\ g_0(x) & g_0(x_n) & \cdots & g_0(x_{n+k-j-1}) \\ \cdots & \cdots & \cdots & \cdots \\ g_{k-1}(x) & g_{k-1}(x_n) & \cdots & g_{k-1}(x_{n+k-j-1}) \end{vmatrix}}{D} \qquad (2.3f)$$

This is equivalent to Neville pattern and the way of computing can be shown by Figure 2.2.

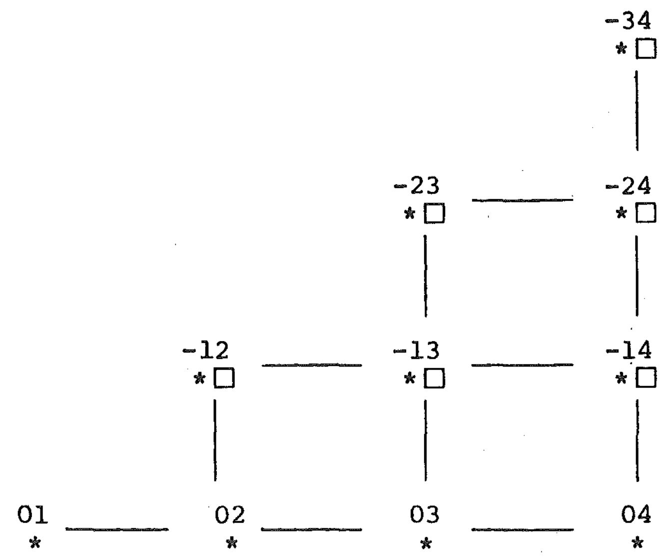



Figure 2.2. * Hg □ Vg

$$\text{Then } P_k^n(x) = P_{k-1}^n(x) - \frac{\Delta P_{k-1}^n(x)}{\Delta Hg_{0,k}^n(x)} Hg_{0,k}^n(x) \tag{2.3g}$$

$$k = 1,2....$$

Alternatively, if we initialize

$$Vg_{k,1-k}^n(x) = \frac{\begin{vmatrix} -g_k(x) & -g_k(x_n) \\ g_0(x) & g_0(x_n) \end{vmatrix}}{g_0(x_n)}$$

$$Hg_{k,1-k}^n(x) = \frac{\begin{vmatrix} -g_k(x) & -g_k(x_n) \\ g_0(x) & g_0(x_n) \end{vmatrix}}{g_0(x_n)} \tag{2.3h}$$

$$k = 1,2,3....$$

then by using (2,3c,d) to compute $Hg^{n}_{k,-j}(x)$ and $Vg^{n}_{k,-j}(x)$ we have

$$P^{n}_{k}(x) = P^{n}_{k-1} - \frac{\Delta P^{n}_{k-1}(x)}{\Delta Vg^{n}_{k,0}(x)} Vg^{n}_{k,0}(x) \qquad (2.3i)$$

and $Vg^{n}_{k,0}(x)$ has the same form as (2.3f).

For linear interpolation, it is better to use the Aitken pattern rather than the Neville since it uses less storage and computation. But for the rational interpolation, Neville pattern is more effective and economical. The ratios of the differences in (2.2) and (2.3b,c) are constant terms as shown in the following lemma.

LEMMA 1:

*The ratio of the differences in* (2.3b) *and* (2.3c) *are constants* ( *independent of* x).

*Proof:*

With the definition of $Hg^{n}_{-(j+1),k}(x)$ and $Hg^{n}_{-j,k-1}(x)$ in (2.3d,e) by using the Sylvester's identity and exchanging some rows or columns, after term eliminating, we have

$$\frac{\Delta\, Hg^{n}_{-(j+1),k}(x)}{\Delta\, Vg^{n}_{-j,k-1}(x)} =$$

$$\left\{ - \begin{vmatrix} g_0(x_n) & g_0(x) & g_0(x_{n+1}) & \cdot & g_0(x_{n+k-j-1}) \\ g_{j+2}(x_n) & g_{j+2}(x) & g_{j+2}(x_{n+1}) & \cdot & g_{j+2}(x_{n+k-j-1}) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ g_{k-1}(x_n) & g_{k-1}(x) & g_{k-1}(x_{n+1}) & \cdot & g_{k-1}(x_{n+k-j-1}) \\ 0 & 1 & 0 & \ldots & 0 \end{vmatrix} \begin{vmatrix} -g_k(x) & -g_k(x_{n+1}) & \cdots & -g_k(x_{n+k-j}) \\ g_0(x) & g_0(x_{n+1}) & \cdots & g_0(x_{n+k-j}) \\ g_{j+2}(x) & g_{j+2}(x_{n+1}) & \cdots & g_{j+2}(x_{n+k-j}) \\ \cdots & \cdots & \cdots & \cdots \\ g_{k-1}(x) & g_{k-1}(x_{n+1}) & \cdots & g_{k-1}(x_{n+k-j}) \end{vmatrix} + \right.$$

$$\left. \begin{vmatrix} g_0(x) & g_0(x_{n+1}) & \cdots & g_0(x_{n+k-j}) \\ g_{j+2}(x) & g_{j+2}(x_{n+1}) & \cdots & g_{j+2}(x_{n+k-j}) \\ \cdots & \cdots & \cdots & \cdots \\ g_{k-1}(x) & g_{k-1}(x_{n+1}) & \cdots & g_{k-1}(x_{n+k-j}) \\ 1 & 0 & \cdots & 0 \end{vmatrix} \begin{vmatrix} -g_k(x_n) & -g_k(x) & -g_k(x_{n+1}) & \cdots & -g_k(x_{n+k-j-1}) \\ g_0(x_n) & g_0(x) & g_0(x_{n+1}) & \cdots & g_0(x_{n+k-j-1}) \\ g_{j+2}(x_n) & g_{j+2}(x) & g_{j+2}(x_{n+1}) & \cdot & g_{j+2}(x_{n+k-j-1}) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ g_{k-1}(x_n) & g_{k-1}(x) & g_{k-1}(x_{n+1}) & \cdot & g_{k-1}(x_{n+k-j-1}) \end{vmatrix} \right\} \Big/$$

$$\left\{ - \begin{vmatrix} g_0(x_n) & g_0(x) & g_0(x_{n+1}) & \cdots & g_0(x_{n+k-j-1}) \\ g_{j+2}(x_0) & g_{j+2}(x) & g_{j+2}(x_{n+1}) & \cdot & g_{j+2}(x_{n+k-j-1}) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ g_{k-1}(x_n) & g_{k-1}(x) & g_{k-1}(x_{n+1}) & \cdot & g_{k-1}(x_{n+k-j-1}) \\ 0 & 1 & 0 & \cdots & 0 \end{vmatrix} \begin{vmatrix} -g_{k-1}(x) & -g_{k-1}(x_{n+1}) & \cdots & -g_{k-1}(x_{n+k-j}) \\ g_0(x) & g_0(x_{n+1}) & \cdots & g_0(x-1+k-j) \\ g_{j+1}(x) & g_{j+1}(x_{n+1}) & \cdots & g_{j+1}(x_{n+k-j}) \\ \cdots & \cdots & \cdots & \cdots \\ g_{k-2}(x) & g_{k-2}(x_{n+1}) & \cdots & g_{k-2}(x_{n+k-j}) \end{vmatrix} + \right.$$

$$\left. \begin{vmatrix} g_0(x) & g_0(x_{n+1}) & \cdots & g_0(x_{n+k-j}) \\ g_{j+2}(x) & g_{k+2}(x_{n+1}) & \cdots & g_{k+2}(x_{n+k-j}) \\ \cdots & \cdots & \cdots & \cdots \\ g_{k-1}(x) & g_{k-1}(x_{n+1}) & \cdots & g_{k-1}(x_{n+k-j}) \\ 1 & 0 & \cdots & 0 \end{vmatrix} \begin{vmatrix} -g_{k-1}(x_n) & -g_{k-1}(x) & -g_{k-1}(x_{n+1}) & \cdot & g_{k-i}(x_{n+k-j-1}) \\ g_0(x_n) & g_0(x) & g_0(x_{n+1}) & \cdots & g_0(x_{n+k-j-1}) \\ g_{j+1}(x_n) & g_{j+1}(x) & g_{j+1}(x_{n+1}) & \cdot & g_{j+1}(x_{n+k-j-1}) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ g_{k-2}(x_n) & g_{k-2}(x) & g_{k-2}(x_{n+1}) & \cdot & g_{k-2}(x_{n+k-j-1}) \end{vmatrix} \right\}$$

$$= (-1)^{k-j} \frac{\begin{vmatrix} -g_k(x) & -g_k(x_n) & \cdots & -g_k(x_{n+k-j}) \\ g_0(x) & g_0(x_n) & \cdots & g_0(x_{n+k-j}) \\ g_{j+2}(x) & g_{j+2}(x_n) & \cdots & g_{j+2}(x_{n+k-j}) \\ \cdots & \cdots & \cdots & \cdots \\ g_{k-1}(x) & g_{k-1}(x_n) & \cdots & g_{k-1}(x_{n+k-j}) \\ 1 & 0 & \cdots & 0 \end{vmatrix}}{\begin{vmatrix} g_{j+1}(x) & g_{j+1}(x_n) & \cdots & g_{j+1}(x_{n+k-j}) \\ g_0(x) & g_0(x_n) & \cdots & g_0(x_{n+k-j}) \\ g_{j+2}(x) & g_{j+2}(x_n) & \cdots & g_{j+2}(x_{n+k-j}) \\ \cdots & \cdots & \cdots & \cdots \\ -g_{k-1}(x) & -g_{k-1}(x_n) & \cdots & -g_{k-1}(x_{n+k-j}) \\ 1 & 0 & \cdots & 0 \end{vmatrix}} = \frac{\begin{vmatrix} -g_k(x_n) & \cdots & -g_k(x_{n+k-j}) \\ g_0(x_n) & \cdots & g_0(x_{n+k-j}) \\ g_{j+2}(x_n) & \cdots & g_{j+2}(x_{n+k-j}) \\ \cdots & \cdots & \cdots \\ g_{k-1}(x_n) & \cdots & g_{k-1}(x_{n+k-j}) \end{vmatrix}}{\begin{vmatrix} -g_{k-1}(x_n) & \cdots & -g_{k-1}(x_{n+k-j}) \\ g_0(x_n) & \cdots & g_0(x_{n+k-j}) \\ g_{j+1}(x_n) & \cdots & g_{j+1}(x_{n+k-j}) \\ \cdots & \cdots & \cdots \\ g_{k-2}(x_n) & \cdots & g_{k-2}(x_{n+k-j}) \end{vmatrix}} = \textit{constant}$$

where $j + 2 < k$

<u>LEMMA 2:</u>

*The ratios of the differences in* (2.3g) *and* (2.3i) *are constants*

*Proof:* Similar to Lemma 1.

It is interesting to observe that $g^n_{k-1,k}(x)$ [ $\nabla g^n_{k,0}(x)$, $Hg^n_{0,k}(x)$] in the algorithm acts to increase the degree of $P^n_k(x)$. The key of this algorithm is to build up $g^n_{k,i}(x)$[ $\nabla g^n_{k,i}(x)$, $Hg^n_{k,i}(x)$] These play an important role in the generalization to the rational interpolating function $R^i_{m,n}(x)$ with $R^i_{m,n}(x_j) = f_j$ where $j = i, i+1 \ldots i+m+n$.

## 3 THE RATIONAL INTERPOLATION ALGORITHM

The interpolating rational function $R^i_{m,n}(x)$ (1.1) may be expressed implicitly in the form (1.2). Then $h_0(x)R^i_{m,n}(x)$ can be expressed in the same form as (2.1) where $f_j$ is replaced by $h_0(x_j)f_j$, $j = i, i+1, \ldots, i+m+n$ and $g_k(x) = h_k(x)$, $k = 0,1,\ldots,m$; $g_{m+\ell}(x) = h_\ell(x)R^i_{m,n}(x)$, $\ell = 1,2,\ldots,n$.

If we assume $g_k(x) = x^k, k = 0,1 \ldots m$, $g_{m+\ell}(x) = x^\ell f(x), \ell = 1, 2 \ldots n$, then $R^i_{m,n}(x)$ can be expressed as below (3.1).

(Clearly a minor notational change will suffice to handle the more general case when $h_0(x)$ is not identically 1).

$$R^i_{m,n}(x) = \frac{\begin{vmatrix} 0 & -f_i & \cdots & -f_{i+m+n} \\ 1 & 1 & \cdots & 1 \\ x & x_i & \cdots & x_{i+m+n} \\ \cdots & \cdots & \cdots & \cdots \\ x^m & x_i^m & \cdots & x^m_{i+m+n} \\ xf & x_i f_i & \cdots & x_{i+m+n} f_{i+m+n} \\ \cdots & \cdots & \cdots & \cdots \\ x^n f & x_i^n f_i & \cdots & x^n_{i+m+n} f_{i+m+n} \end{vmatrix}}{D} \qquad (3.1)$$

We can build up $g^i_{m-1,m}(x)$ where $m = 1,2 \ldots$ for computing $R^i_{m,0}(x)$, and $g^i_{n-1,n}(x)$ where $n = 1,2 \ldots$ for computing $R^i_{0,n}(x)$ by using (2.2) which is based on Aitken's pattern, or build up $Vg^i_{m,0}(x)$ for $R^i_{m,0}(x)$ and $Hg^i_{0,n}(x)$for $R^i_{0,n}(x)$by using (2.3) which is based on Neville pattern. Then by using the relation of $Vg^i_{m,0}(x)$ $(g^i_{m-1,m}(x))$ and $Hg^i_{0,n}(x)$ $(g^i_{n-1,n}(x))$ we build up

$Vg^{i}_{m,n}(x)$ and $Hg^{i}_{m,n}(x)$ which is based on Neville's pattern for constructing $R^{i}_{m,n}(x)$ either from the previous column or row. The following algorithm is based on the Neville pattern.

1. Initialization.

For m,n = 1,2,... N and for i = 0,1,...,N-1

$$Vg^{i}_{m,1-m}(x) = \frac{\begin{vmatrix} -g_m(x) & -g_m(x_i) \\ g_0(x) & g_0(x_i) \end{vmatrix}}{g_0(x_i)}$$

$$Hg^{i}_{m,1-m}(x) = \frac{\begin{vmatrix} -g_m(x) & -g_m(x_i) \\ g_0(x) & g_0(x_i) \end{vmatrix}}{g_0(x_i)} \qquad (3.2a)$$

where $g_0(x_i) = 1$ $\quad m = 1,2 \ldots.$

$g_m(x_i) = x_i^m$ $\quad i = 0,1 \ldots.$

$$Vg^{i}_{1-n,n}(x) = \frac{\begin{vmatrix} -g_n(x) & -g_n(x_i) \\ g_0(x) & g_0(x_i) \end{vmatrix}}{g_0(x_i)}$$

$$Hg^{i}_{1-n,n}(x) = \frac{\begin{vmatrix} -g_n(x) & -g_n(x_i) \\ g_0(x) & g_0(x_i) \end{vmatrix}}{g_0(x_i)} \qquad (3.2b)$$

where $g_n(x_i) = x_i^n f_i$

$g_0(x_i) = 1$ $\quad n = 1,2 \ldots$

2. For $k = 2,3,\ldots,N$

For $m = N, N-1,\ldots,k-N$

Set $n = k - m$

For $i = 0,1,\ldots,N-k$

$$Hg^{i}_{m,n}(x) = Hg^{i}_{m-1,n}(x) - \frac{\Delta\, Hg^{i}_{m-1,n}(x)}{\Delta\, Vg^{i}_{m,n-1}(x)} Vg^{i}_{m,n-1}(x) \qquad (3.2c)$$

$$Vg^{i}_{m,n}(x) = Vg^{i}_{m,n-1}(x) - \frac{\Delta\, Vg^{i}_{m,n-1}(x)}{\Delta\, Hg^{i}_{m-1,n}(x)} Hg^{i}_{m-1,n}(x) \qquad (3.2d)$$

3. For $i = 0, 1,\ldots,N$, define $R^{i}_{0,0}(x) = f_i$

For $n = 1,2,\ldots,N$

For $i = 0,1,\ldots,N-n$

$$R^{i}_{0,n}(x) = R^{i}_{0,n-1}(x) - \frac{\Delta R^{i}_{0,n-1}(x)}{\Delta Hg^{i}_{0,n}(x)} Hg^{i}_{0,n}(x).$$

For $m = 1,2,\ldots,N$

For $i = 0,1,\ldots,N-m$

$$R^{i}_{m,0}(x) = R^{i}_{m-1,0}(x) - \frac{\Delta R^{i}_{m-1,0}(x)}{\Delta Vg^{i}_{m,0}(x)} Vg^{i}_{m,0}(x)$$

For $m,n = 1,2,\ldots N$

For $i = 0,1,\ldots,N-m-n$

$$R^{i}_{m,n}(x) = R^{i}_{m,n-1}(x) - \frac{\Delta\, R^{i}_{m,n-1}(x)}{\Delta\, Hg^{i}_{m,n}(x)} Hg^{i}_{m,n}(x) \qquad (3.2e)$$

$$R^{i}_{m,n}(x) = R^{i}_{m-1,n}(x) - \frac{\Delta\, R^{i}_{m-1,n}(x)}{\Delta\, Vg^{i}_{m,n}(x)} Vg^{i}_{m,n}(x) \qquad (3.2f)$$

The algorithm follows from the relations proved in the following Theorem.

THEOREM 1

*Equations* (3.2e) *and* (3.2f) *hold for* $m,n = 0,1,2\ldots\ldots$ *provided* $n > 1$ *for* (3.2e) *and* $m > 1$ *for* (3.2f).

*Proof:* To prove (3.2e), using the Sylvester's identity for the determinants, $R^{i}_{m,n}(x)$ can be decomposed as the following:

$$
R^{i}_{m,n}(x) = \frac{\begin{vmatrix}
0 & -f_i & \ldots & -f_{i+m+n-1} \\
1 & 1 & \ldots & 1 \\
x & x_i & \ldots & x_{i+m+n-1} \\
\cdots & \cdots & \cdots & \cdots \\
x^m & x_i^m & \ldots & x^m_{i+m+n-1} \\
xf & x_i f_i & \ldots & x_{i+m+n-1} f_{i+m+n-1} \\
\cdots & \cdots & \cdots & \cdots \\
x^{n-1} f & x_i^{n-1} f_i & \ldots & x_{i+m+n-1} f_{i+m+n-1}
\end{vmatrix}}{D} -
$$

$$
\frac{\begin{vmatrix}
-f_i & \ldots & -f_{i+m+n} \\
1 & \ldots & 1 \\
x_i & \ldots & x_{i+m+n} \\
\cdots & \cdots & \cdots \\
x_i^m & \ldots & x^m_{i+m+n} \\
x_i f_i & \ldots & x_{i+m+n} f_{i+m+n} \\
\cdots & \cdots & \cdots \\
x_i^{n-1} f_i & \ldots & x^{n-1}_{i+m+n} f_{i+m+n}
\end{vmatrix}
\begin{vmatrix}
1 & & \ldots & 1 \\
x & x_i & \ldots & x_{i+m+n-1} \\
\cdots & \cdots & \cdots & \cdots \\
x^m & x_i^m & \ldots & x^m_{i+m+n-1} \\
xf & x_i f_i & \ldots & x_{i+m+n-1} f_{i+m+n-1} \\
\cdots & \cdots & \cdots & \cdots \\
x^n f & x_i^n f_i & \ldots & x^n_{i+m+n-1} f_{i+m+n-1}
\end{vmatrix}}{\begin{vmatrix}
1 & \ldots & 1 \\
x_i & \ldots & x_{i+m+n} \\
\cdots & \cdots & \cdots \\
x_i^m & \ldots & x^m_{i+m+n} \\
x_i f_i & \ldots & x_{i+m+n} f_{i+m+n} \\
\cdots & \cdots & \cdots \\
x_i^n f_i & \ldots & x^n_{i+m+n} f_{i+m+n}
\end{vmatrix}
\begin{vmatrix}
1 & \ldots & 1 \\
x_i & \ldots & x_{i+m+n-1} \\
\cdots & \cdots & \cdots \\
x_i^m & \ldots & x^m_{i+m+n-1} \\
x_i f_i & \ldots & x_{i+m+n-1} f_{i+m+n-1} \\
\cdots & \cdots & \cdots \\
x_i^{n-1} f_i & \ldots & x^{n-1}_{i+m+n-1} f_{i+m+n-1}
\end{vmatrix}}
$$

We define

$$Hg^{i}_{m,n}(x) = \frac{\begin{vmatrix} -x^n f & -x_i^n f_i & \cdots & -x^n_{i+m+n-1} f_{i+m+n-1} \\ 1 & 1 & \cdots & 1 \\ x & x_i & \cdots & x_{i+m+n-1} \\ \cdots & \cdots & \cdots & \cdots \\ x^m & x_i^m & \cdots & x^m_{i+m+n-1} \\ xf & x_i f_i & \cdots & x_{i+m+n-1} f_{i+m+n-1} \\ \cdots & \cdots & \cdots & \cdots \\ x^{n-1} f & x_i^{n-1} f_i & \cdots & x^{n-1}_{i+m+n-1} f_{i+m+n-1} \end{vmatrix}}{\begin{vmatrix} 1 & \cdots & 1 \\ x_i & \cdots & x_{i+m+n-1} \\ \cdots & \cdots & \cdots \\ x_i^m & \cdots & x^m_{i+m+n-1} \\ x_i f_i & \cdots & x_{i+m+n-1} f_{i+m+n-1} \\ \cdots & \cdots & \cdots \\ x_i^{n-1} f_i & \cdots & x^{n-1}_{i+m+n-1} f_{i+m+n-1} \end{vmatrix}} \tag{3.2g}$$

which is the second factor of the second term by shifting the last row to the first row and change the sign. The first factor of the second term is in fact equal to $\dfrac{\Delta\, R^{i}_{m,n-1}(x)}{\Delta\, Hg^{i}_{m,n}(x)}$ .

In the same way as in lemma 1, it can be shown that

$$\frac{\Delta R^{i}_{m,n-1}(x)}{\Delta Hg^{i}_{m,n}(x)} = \frac{\begin{vmatrix} -f_i & \cdots & -f_{i+m+n} \\ 1 & \cdots & 1 \\ x_i & \cdots & x_{i+m+n} \\ \cdots & \cdots & \cdots \\ x_i^m & \cdots & x_{i+m+n}^m \\ x_i f_i & \cdots & x_{i+m+n} f_{i+m+n} \\ \cdots & \cdots & \cdots \\ x_i^{n-1} f_i & \cdots & x_{i+m+n}^{n-1} f_{i+m+n} \end{vmatrix}}{\begin{vmatrix} -x_i^n f_i & \cdots & -x_{i+m+n}^{n-1} f_{i+m+n} \\ 1 & \cdots & 1 \\ x_i & \cdots & x_{i+m+n} \\ \cdots & \cdots & \cdots \\ x_i^m & \cdots & x_{i+m+n}^m \\ x_i f_i & \cdots & x_{i+m+n} f_{i+m+n} \\ \cdots & \cdots & \cdots \\ x_i^n f_i & \cdots & x_{i+m+n}^{n-1} f_{i+m+n} \end{vmatrix}}$$

which is a constant formed by the interpolating set $\{(x_j, f_j),\ j = i, \ldots i+m+n\}$.

To prove (3.2f), shift the mth row of the numerator in (3.1) to the last row and the proof follows using the same method as above, where in this case the second factor is defined to be $Vg^{i}_{m,n}(x)$.

Where we define

$$Vg^{i}_{m,n}(x) = \frac{\begin{vmatrix} -x^n f & -x_i^n f_i & \cdots & -x^n_{i+m+n-1} f_{i+m+n-1} \\ 1 & 1 & \cdots & 1 \\ x & x_i & \cdots & x_{i+m+n-1} \\ \cdots & \cdots & \cdots & \cdots \\ x^m & x_i^m & \cdots & x^m_{i+m+n-1} \\ xf & x_i f_i & \cdots & x_{i+m+n-1} f_{i+m+n-1} \\ \cdots & \cdots & \cdots & \cdots \\ x^{n-1} f & x_i^{n-1} f_i & \cdots & x^{n-1}_{i+m+n-1} f_{i+m+n-1} \end{vmatrix}}{\begin{vmatrix} 1 & \cdots & 1 \\ x_i & \cdots & x_{i+m+n-1} \\ \cdots & \cdots & \cdots \\ x_i^{m-1} & \cdots & x^{m-1}_{i+m+n-1} \\ x_i f_i & \cdots & x_{i+m+n-1} f_{i+m+n-1} \\ \cdots & \cdots & \cdots \\ x_i^n f_i & \cdots & x^n_{i+m+n-1} f_{i+m+n-1} \end{vmatrix}} \tag{3.2h}$$

Note: 1. The expressions for $Hg^{i}_{m,n}(x)$ and $Vg^{i}_{m,n}(x)$ have common numerators and the denominators have one row different.

2. The expressions for $Hg^{i}_{m-1,n}(x)$ and $Vg^{i}_{m,n-1}(x)$ have the same denominators but the numerators have one row different.

THEOREM 2.

*Equations* (3.2c) *and* (3.2d) *hold for all integers* m,n *such that* $m + n \geq 1$.

*Proof:* Using the same method in Theorem 1, we obtain the above results.

COROLLARY 1.

*The ratios of the differences in* (3.2c), (3.2d), (3.2e) *and* (3.2f) *are constants (independent of $x$).*

*Proof*:

Consider $\dfrac{\Delta R^i_{m,n-1}(x)}{\Delta Hg^i_{m,n}(x)}$, the ratio of the differences in (3.2e). From the proof of Theorem 1 it is clear that the ratio is a constant term ( independent of x ). The proof for the other ratios follows similarly.

COROLLARY 2.

$$Hg^i_{m,n}(x) = k_i . Vg^i_{m,n}(x) \quad \textit{where } k_i = -\frac{\Delta Hg^i_{m-1,n}(x)}{\Delta Vg^i_{m,n-1}(x)}$$

*Proof*:

This follows immediately by combining (3.2c) and (3.2d) since the ratios of the differences are inverses and, by Corollary 1, constant.

COROLLARY 3.

*In the case* $h_j(x) = x^j$ , *the maximum degree*, $k$, *of the terms* $x^k$ *and* $x^k R^i_{m,n}(x)$ *in* $Vg^i_{m,n}(x)$ *and* $Hg^i_{m,n}(x)$ *are* $m$ *and* $n$ *respectively*.

*Proof*:

From the initialization (3.2a,b) it is clear that the degree of $x^k$ in $Vg^i_{m,1-m}(x) = Hg^i_{m,1-m}(x)$ is m and the degree, $k$, of $x^k R$ in $Vg^i_{1-n,n}(x) = Hg^i_{1-n,n}(x)$ is n. By induction using (3.2c,d) and Corollary 1, the maximum degree of $x^k$ in $Vg^i_{m,n}(x)$ and $Hg^i_{m,n}(x)$ is at most m. Similarly for $x^k R$

COROLLARY 4. *If* $Hg^{i}_{m,n}(x) = Hg^{i+1}_{m,n}(x)$, *then* $k_i = k_{i+1}$ *(where* $k_i$ *has been defined in Corollary 2) and* $Vg^{i}_{m,n}(x) = Vg^{i+1}_{m,n}(x)$.

*Proof:* Since $Hg^{i}_{m,n}(x) = Hg^{i+1}_{m,n}(x)$

$$\frac{Hg^{i}_{m-1,n}(x)Vg^{i+1}_{m,n-1}(x) - Hg^{i+1}_{m-1,n}(x)Vg^{i}_{m,n-1}(x)}{Vg^{i+1}_{m,n-1}(x) - Vg^{i}_{m,n-1}(x)}$$

$$= \frac{Hg^{i+1}_{m-1,n}(x)Vg^{i+2}_{m,n-1}(x) - Hg^{i+2}_{m-1,n}(x)Vg^{i+1}_{m,n-1}(x)}{Vg^{i+2}_{m,n-1}(x) - Vg^{i+1}_{m,n-1}(x)}$$

By cross multiplying and inserting $- Vg^{i+1}_{m,n-1}(x)Hg^{i+1}_{m-1,n}(x)$ in both sides, it follows

$$\Delta\, Hg^{i}_{m-1,n}(x)\Delta\, Vg^{i+1}_{m,n-1}(x) = \Delta\, Hg^{i+1}_{m-1,n}(x)\Delta\, Vg^{i}_{m,n-1}(x).$$

Hence $k_i = k_{i+1}$ and by Corollary 2 it follows $Vg^{i}_{m,n}(x) = Vg^{i+1}_{m,n}(x)$.

Note. All these results follow in similar way for the general case when $x_i^k$ is replaced by $h_k(x_i)$ and $x_i^k f_i$ replaced by $h_k(x_i)f_i$.

## 4. THE SINGULAR CASE

These algorithms may break down if the adjacent terms are equal. For example $\Delta\, Hg^{i}_{m,n-1}(x)$ is zero if $Hg^{i+1}_{m,n-1}(x) = Hg^{i}_{m,n-1}(x)$.

Then since $Vg^{i+1}_{m,n-1}(x) = Vg^{i}_{m,n-1}(x)$ from Corollary 4, it follows

$Hg^{i}_{m,n}(x) = \infty$ and $Vg^{i}_{m+1,n-1}(x) = \infty$.

In this discussion it is assumed that such singularities are isolated [cf 11] and correspond loosely to the semi-normal case in Pade approximation. This assumption will be satisfied if the set of given functions form a quasicomplete Chebyshev system on the set of interpolating points [7].

In the case noted above, the indeterminancy of $Hg^{i}_{m,n}(x)$ implies that $\Delta Hg^{j}_{m,n}(x)$ and hence $Hg^{j}_{m+1,n}(x)$ for $j = i-1, i$ cannot be computed. This difficulty may be overcome by jumping two steps. Instead of (3.2c) it can be shown

$$Hg^{i}_{m+1,n}(x) = Hg^{i-1}_{m,n}(x) - \Delta g'.Vg^{i-1}_{m+1,n-1}(x) \qquad (4.1a)$$

$$Hg^{i}_{m+1,n}(x) = Hg^{i+1}_{m,n}(x) - \Delta g.Vg^{i+1}_{m+1,n-1}(x) \qquad (4.1b)$$

where if $m = N, N-1, \ldots\ldots k-n$ and $m+n = k$

$$Hg^{i-1}_{m+1,n}(x) = Hg^{i-1}_{m,n}(x) - \frac{\Delta Hg^{i-1}_{m,n}(x)}{\Delta Vg^{i-1}_{m+1,n-1}(x)} Vg^{i-1}_{m+1,n-1}(x)$$

where

$$\Delta g' = \frac{\Delta Hg^{i-1}_{m,n}(x)}{\Delta Vg^{i-1}_{m+1,n-1}(x)} = \frac{Vg^{i}_{m,n-1}(x)\left(\dfrac{\Delta Hg^{i}_{m-1,n}(x)}{\Delta Vg^{i}_{m,n-1}(x)} - \dfrac{\Delta Hg^{i-1}_{m-1,n}(x)}{\Delta Vg^{i-1}_{m,n-1}(x)}\right)}{Hg^{i}_{m,n-1}(x)\left(\dfrac{\Delta Vg^{i}_{m+1,n-2}(x)}{\Delta Hg^{i}_{m,n-1}(x)} - \dfrac{\Delta Vg^{i-1}_{m+1,n-2}(x)}{\Delta Hg^{i-1}_{m,n-1}(x)}\right)}$$

If $Vg^{i}_{m,n-1}(x) = Hg^{i}_{m,n-1}(x)$ for all $i$. Then

$$\Delta g' = \frac{\Delta Hg^{i}_{m-1,n}(x)\ \Delta Vg^{i-1}_{m,n-1}(x) - \Delta Hg^{i-1}_{m-1,n}(x)\ \Delta Vg^{i}_{m,n-1}(x)}{\Delta Vg^{i}_{m+1,n-2}(x)\ \Delta Hg^{i-1}_{m,n-1}(x) - \Delta Vg^{i-1}_{m+1,n-2}(x)\ \Delta Hg^{i}_{m,n-1}(x)}$$

When $\Delta Hg^{i}_{m,n-1}(x) = \Delta Vg^{i}_{m,n-1}(x) = 0$, the above form can be reduced to

$$\frac{\Delta Hg^{i-1}_{m,n}(x)}{\Delta Vg^{i-1}_{m+1,n-1}(x)} = \frac{\Delta Hg^{i}_{m-1,n}(x)}{\Delta Vg^{i}_{m+1,n-2}(x)}$$

By the same method it can be shown

$$\Delta g = \Delta g' = \frac{\Delta\, Hg^{i}_{m-1,n}(x)}{\Delta\, Vg^{i}_{m+1,n-2}(x)} \tag{4.2a}$$

This applies to the case when the singularity occurs at the first step. i.e. k = 2, otherwise more work is required to derive $\Delta g = \dfrac{\Delta\, Hg^{i}_{m,n}(x)}{\Delta\, Vg^{i}_{m+1,n-1}(x)}$ as follows:

$$\frac{\Delta\, Hg^{i}_{m,n}(x)}{\Delta\, Vg^{i}_{m+1,n-1}(x)} = \frac{Vg^{i+1}_{m,n-1}(x)}{Hg^{i+1}_{m,n-1}(x)}$$

$$= \frac{\Delta\, Hg^{i+1}_{m-1,n}(x)\ \Delta\, Vg^{i}_{m,n-1}(x) - \Delta\, Hg^{i}_{m-1,n}(x)\ \Delta\, Vg^{i+1}_{m,n-1}(x)}{\Delta\, Vg^{i+1}_{m+1,n-2}(x)\ \Delta\, Hg^{i}_{m,n-1}(x) - \Delta\, Vg^{i}_{m+1,n-2}(x)\ \Delta\, Hg^{i+1}_{m,n-1}(x)}$$

$$= \frac{\Delta\, Hg^{i}_{m,n-1}(x)\ \Delta\, Hg^{i+1}_{m,n-1}(x)}{\Delta\, Vg^{i}_{m,n-1}(x)\ \Delta\, Vg^{i+1}_{m,n-1}(x)}$$

where $\dfrac{\Delta\, Hg^{i}_{m,n-1}(x)\ \Delta\, Hg^{i+1}_{m,n-1}(x)}{\Delta\, Vg^{i}_{m,n-1}(x)\ \Delta\, Vg^{i+1}_{m,n-1}(x)}$

$$= \frac{Vg^{i+1}_{m,n-2}(x)\left(\dfrac{\Delta\, Hg^{i}_{m-1,n-1}(x)}{\Delta\, Vg^{i}_{m,n-2}(x)} - \dfrac{\Delta\, Hg^{i+1}_{m-1,n-1}(x)}{\Delta\, Vg^{i+1}_{m,n-2}(x)}\right)}{Hg^{i+1}_{m-1,n-1}(x)\left(\dfrac{\Delta\, Vg^{i}_{m,n-2}(x)}{\Delta\, Hg^{i}_{m-1,n-1}(x)} - \dfrac{\Delta\, Vg^{i+1}_{m,n-2}(x)}{\Delta\, Hg^{i+1}_{m-1,n-1}(x)}\right)}$$

$$\times \frac{Vg^{i+2}_{m,n-2}(x)\left(\dfrac{\Delta\, Hg^{i+1}_{m-1,n-1}(x)}{\Delta\, Vg^{i+1}_{m,n-2}(x)} - \dfrac{\Delta\, Hg^{i+2}_{m-1,n-1}(x)}{\Delta\, Vg^{i+2}_{m,n-2}(x)}\right)}{Hg^{i+2}_{m-1,n-1}(x)\left(\dfrac{\Delta\, Vg^{i+1}_{m,n-2}(x)}{\Delta\, Hg^{i+1}_{m-1,n-1}(x)} - \dfrac{\Delta\, Vg^{i+2}_{m,n-2}(x)}{\Delta\, Hg^{i+2}_{m-1,n-1}(x)}\right)}$$

$$= \frac{Vg^{i+1}_{m,n-2}(x)}{Hg^{i+1}_{m-1,n-1}(x)} \cdot \frac{Vg^{i+2}_{m,n-2}(x)}{Hg^{i+2}_{m-1,n-1}(x)} \cdot \frac{\Delta Hg^{i}_{m-1,n-1}(x)}{\Delta Vg^{i}_{m,n-2}(x)}$$

$$\left(\frac{\Delta Hg^{i+1}_{m-1,n-1}(x)}{\Delta Vg^{i+1}_{m,n-2}(x)}\right)^2 \cdot \frac{\Delta Hg^{i+2}_{m-1,n-1}(x)}{\Delta Vg^{i+2}_{m-1,n-2}(x)}$$

If $\Delta Vg^{i}_{m,n-1}(x) = \Delta Hg^{i}_{m,n-1}(x) = 0$, then

$$\frac{\Delta Hg^{i}_{m,n}(x)}{\Delta Vg^{i}_{m+1,n-1}(x)} = -\frac{\Delta Vg^{i+1}_{m,n-2}(x)}{\Delta Hg^{i+1}_{m-1,n-1}(x)} \cdot \frac{\Delta Hg^{i}_{m-1,n}(x)}{\Delta Vg^{i}_{m+1,n-2}(x)} \cdot \frac{\Delta Vg^{i+1}_{m,n-1}(x)}{\Delta Hg^{i+1}_{m,n-1}(x)}$$

$$\frac{Vg^{i+1}_{m,n-2}(x)}{Hg^{i+1}_{m-1,n-1}(x)} \cdot \frac{Vg^{i+2}_{m,n-2}(x)}{Hg^{i+2}_{m-1,n-1}(x)} \cdot \frac{\Delta Hg^{i}_{m-1,n-1}(x)}{\Delta Vg^{i}_{m,n-2}(x)} \cdot$$

$$\left(\frac{\Delta Hg^{i+1}_{m-1,n-1}(x)}{\Delta Vg^{i+1}_{m,n-2}(x)}\right)^2 \cdot \frac{\Delta Hg^{i+2}_{m-1,n-1}(x)}{\Delta Vg^{i+2}_{m,n-2}(x)}$$

After the terms rearrangment, these can be expressed as

$$\Delta g = -\Delta_1 . \Delta_2 . \Delta_3 \qquad (4.2b)$$

where

$$\Delta_1 = \frac{Vg^{i+1}_{m,n-2}(x)}{\Delta Vg^{i}_{m+1,n-2}(x)}$$

$$\Delta_2 = \frac{\Delta Hg^{i}_{m-1,n}(x)}{Hg^{i+1}_{m-1,n-1}(x)} \cdot \frac{\Delta Hg^{i}_{m-1,n-1}(x)}{\Delta Vg^{i}_{m,n-2}(x)} \cdot \frac{\Delta Hg^{i+1}_{m-1,n-1}(x)}{\Delta Vg^{i+1}_{m,n-2}(x)}$$

$$\Delta_3 = \frac{\Delta Vg^{i+1}_{m,n-1}(x)}{Hg^{i+2}_{m-1,n-1}(x)} \cdot \frac{Vg^{i+2}_{m,n-2}(x)}{\Delta Hg^{i+1}_{m,n-1}(x)} \cdot \frac{\Delta Hg^{i+2}_{m-1,n-1}(x)}{\Delta Vg^{i+2}_{m,n-2}(x)}$$

and by the same method,

$$\Delta'_g = -\Delta_1 . \Delta_2 . \Delta'_3 \qquad (4.2c)$$

where

$$\Delta'_3 = \frac{\Delta\, Vg^{i-1}_{m,n-1}(x)}{Hg^{i}_{m-1,n-1}(x)} \cdot \frac{Vg^{i}_{m,n-2}(x)}{\Delta\, Hg^{i-1}_{m,n-1}(x)} \cdot \frac{\Delta\, Hg^{i-1}_{m-1,n-1}(x)}{\Delta\, Vg^{i-1}_{m,n-2}(x)}$$

if $k = 3,4,\ldots N$

The computation of $Vg^{j}_{m+1,n}(x)$ , $j = i-1,i$ may be similarly accomplished by replacing (3.2d) by formulas analogous to (4.1) with the terms $\Delta g$ and $\Delta'_g$ replaced by $1/\Delta g$ and $1/\Delta'_g$ respectively.

All these factors can be shown by the following corollary to be constant terms, and their relation can be shown by Figures 4.1 and 4.2.

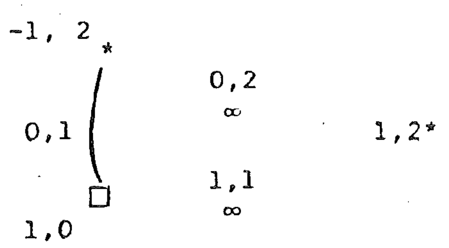
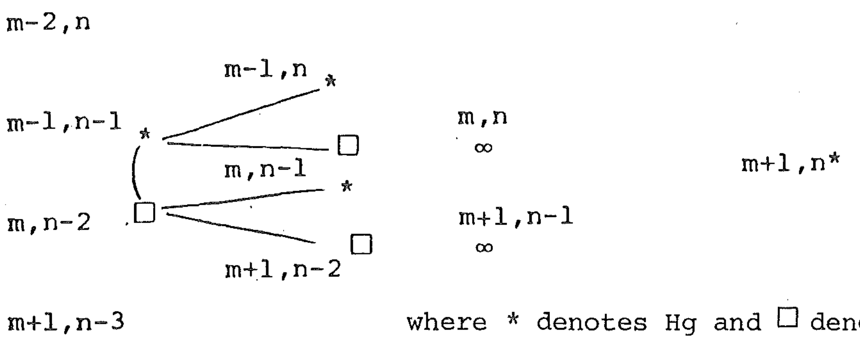



Figure 4.1

Singularity occurs

when $\Delta\, Hg^{i}_{1-k,k}(x) = \Delta\, Vg^{i}_{1-k,k}(x) = 0$ for $k = 1,2\ldots$, e.g. $k = 1$

( or $\Delta\, Hg^{i}_{k,1-k}(x) = \Delta\, Vg^{i}_{k,1-k}(x) = 0$ )




where * denotes Hg and □ denotes Vg.

Figure 4.2.

Singularity occurs

$$\text{when} \quad \Delta\, Hg^{i}_{m,n-1}(x) = \Delta\, Vg^{i}_{m,n-1}(x) = 0$$

If $\Delta Vg^{i}_{0,n-1}(x) = 0$ , then $Hg^{i}_{0,n}(x)$ is indeterminate and hence $\Delta Hg^{j}_{0,n}(x)$ and $R^{j}_{0,n}(x)$, $j = i-1,1$, cannot be computed. By using the analogue of (4.1) and (4.2),

$$R^{i-1}_{0,n}(x) = R^{i-1}_{0,n-1}(x) - \Delta'_{R}.Hg^{i-1}_{0,n}(x) \qquad (4.3a)$$

$$R^{i}_{0,n}(x) = R^{i+1}_{0,n-1}(x) - \Delta_{R}.Hg^{i+1}_{0,n}(x) \qquad (4.3b)$$

where

$$\Delta_R = \Delta'_R = \frac{\Delta R^{i}_{0,n-2}(x)}{\Delta Hg^{\;i}_{-1,n}(x)} \qquad \text{if } n = 2, \text{ and with } m = 0,$$

$$\Delta_R = -\frac{\Delta'_1}{\Delta_2.\Delta_3} \qquad \text{and}$$

$$\Delta'_R = -\frac{\Delta'_1}{\Delta_2.\Delta'_3} \qquad \text{if } n = 3,4,\ldots, \text{ and with } m = 0,$$

where

$$\Delta'_1 = \frac{\Delta R^{i}_{0,n-2}(x)}{Vg^{i+1}_{0,n-2}(x)}$$

Figure 4. 3 shows how the calculation of $R^{i}_{0,n}(x)$ involves $R^{i}_{0,n-2}(x)$, Hg and Vg for the connected points in the way.

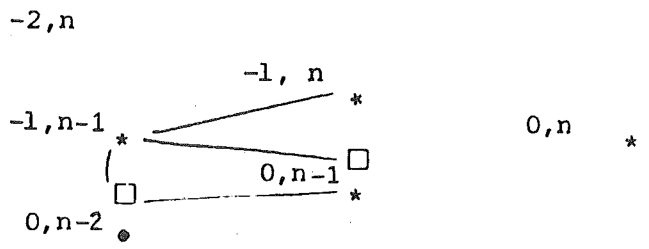



Figure 4.3     * Hg □ Vg ● R

Similarly, if $\Delta Hg^{i}_{m-1,0}(x) = 0$, then $R^{j}_{m,0}(x)$, $j = i-1,i$ may be computed by formulas similar to (4.3)

$$R^{i-1}_{m,0}(x) = R^{i-1}_{m-1,0}(x) - \Delta'_{R}.Vg^{i-1}_{m,0}(x) \qquad (4.4a)$$

$$R^{i}_{m,0}(x) = R^{i+1}_{m-1,0}(x) - \Delta_{R}.Vg^{i+1}_{m,0}(x) \qquad (4.4b)$$

where

$$\Delta_R = \Delta'_R = \frac{\Delta R^{i}_{m-2,0}(x)}{\Delta Vg^{i}_{m,-1}(x)} \qquad \text{if } m = 2, \text{ and with } n = 0,$$

$$\Delta_R = -\frac{\Delta''_1}{\bar{\Delta}_2.\bar{\Delta}_3} \qquad \text{and}$$

$$\Delta'_R = -\frac{\Delta''_1}{\bar{\Delta}_2.\bar{\Delta}'_3} \qquad \text{if } m = 3,4,\ldots, \text{ and with } n = 0,$$

where

$$\Delta''_1 = \frac{\Delta R^{i}_{m-2,0}(x)}{Hg^{i+1}_{m-2,0}(x)}$$

and $\bar{\Delta}_2, \bar{\Delta}_3, \bar{\Delta}'_3$ may be expressed in the same way as the corresponding expression in (4.2b,c) except that $Hg^{i}_{a,b}(x)$ (resp. $Vg^{i}_{a,b}(x)$) is replaced by $Vg^{i}_{b,a}(x)$ (resp. $Hg^{i}_{b,a}(x)$).

Figure 4.4 shows how the calculation of $R^{i}_{m,0}(x)$ involves $R^{i}_{m-2,0}(x)$, Hg and Vg for the connected points in the way.

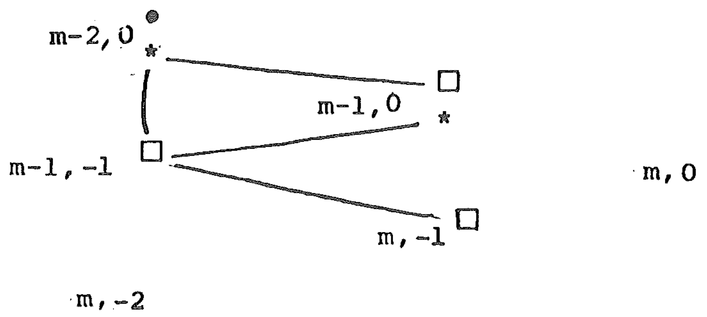



Figure 4.4

In fact the above expressions (4.3) and (4.4) can be generalized to compute $R^{i}_{m,n}(x)$ for m,n = 1,2,3... where some of $R^{i}_{m,n}(x)$ do not exist. But in this case $R^{i}_{m,n}(x)$ may be generated either 'horizontally' or 'vertically' (3.2e,f). If the calculation is blocked one way then it is possible to continue in the other way. Thus the situation is simpler than (4.3) and (4.4)

This approach was motivated by the method used by Wynn [11] to deal with the singular cases.

<u>COROLLARY 5</u>.

*For all m,n, the ratios $\Delta g, \Delta g', \Delta_R, \Delta_R'$ are constants (independent of x)*

[Note that (i) $Vg^{i}_{0,0}(x) = Hg^{i}_{0,0}(x) = 0$

(ii) $Vg^{i}_{m,n}(x)$ $Hg^{i}_{m,n}(x)$ are not defined for $|m| \geqslant n$ if $m < 0$ or for $|n| \geqslant m$ if $n < 0$.

(iii) $R^{i}_{m,n}(x)$ are not defined for $m,n < 0$. ]

*Proof:*

Only the first ratio ( i.e. $\dfrac{\Delta\, Hg^{i}_{m-1,n}(x)}{Hg^{i+1}_{m-1,n-1}(x)}$ ) in $\Delta_2$ of (4.2b) for m and n $\geqslant 0$ is proved. The others follow in a similar way.

From (3.2g,h),

$$\frac{\Delta\, Hg^{i}_{m-1,n}(x)}{Hg^{i+1}_{m-1,n-1}(x)} = \frac{\dfrac{\begin{vmatrix} -x^{n}f & -x^{n}_{i+1}f_{i+1} & \cdots & -x^{n}_{i+m+n-1}f_{i+m+n-1} \\ 1 & 1 & \cdots & 1 \\ x & x_{i+1} & \cdots & x_{i+m+n-1} \\ \cdots & \cdots & \cdots & \cdots \\ x^{m-1} & x^{m-1}_{i+1} & \cdots & x^{m-1}_{i+m+n-1} \\ xf & x_{i+1}f_{i+1} & \cdots & x_{i+m+n-1}f_{i+m+n-1} \\ \cdots & \cdots & \cdots & \cdots \\ x^{n-1}f & x^{n-1}_{i+1}f_{i+1} & \cdots & x^{n-1}_{i+m+n-1}f_{i+m+n-1} \end{vmatrix}}{D^{i+1}_{m-1,n}} \quad \dfrac{\begin{vmatrix} -x^{n}f & -x^{n}_{i}f_{i} & \cdots & -x^{n}_{i+m+n-2}f_{i+m+n-2} \\ 1 & 1 & \cdots & 1 \\ x & x_{i} & \cdots & x_{i+m+n+2} \\ \cdots & \cdots & \cdots & \cdots \\ x^{m-1} & x^{m-1}_{i} & \cdots & x^{m-1}_{i+m+n-2} \\ xf & x_{i}f_{i} & \cdots & x_{i+m+n-2}f_{i+m+n-2} \\ \cdots & \cdots & \cdots & \cdots \\ x^{n-1}f & x^{n-1}_{i}f_{i} & \cdots & x^{n-1}_{i+m+n-2}f_{i+m+n-2} \end{vmatrix}}{D^{i}_{m-1,n}}}{\dfrac{\begin{vmatrix} -x^{n-1}f & -x^{n-1}_{i+1}f_{i+1} & \cdots & -x^{n-1}_{i+m+n-2}f^{n-1}_{i+m+n-2} \\ 1 & 1 & \cdots & 1 \\ x & x_{i+1} & \cdots & x_{i+m+n-2} \\ \cdots & \cdots & \cdots & \cdots \\ x^{m-1} & x^{m-1}_{i+1} & \cdots & x^{m-1}_{i+m+n-2} \\ xf & x_{i+1}f_{i+1} & \cdots & x_{i+m+n-2}f_{i+m+n-2} \\ \cdots & \cdots & \cdots & \cdots \\ x^{n-2}f & x^{n-2}_{i+1}f_{i+1} & \cdots & x^{n-2}_{i+m+n-2}f_{i+m+n-2} \end{vmatrix}}{D^{i+1}_{m-1,n-1}}}$$

$$\text{where } D^{i}_{m-1,n} = \begin{vmatrix} 1 & \cdots & 1 \\ x_i & \cdots & x_{i+m+n-2} \\ \cdots & \cdots & \cdots \\ x^{m-1}_{i} & \cdots & x^{m-1}_{i+m+n-2} \\ x_i f_i & \cdots & x_{i+m+n-2}f_{i+m+n-2} \\ \cdots & \cdots & \cdots \\ x^{n-1}_{i}f_i & \cdots & x^{n-1}_{i+m+n-2}f_{i+m+n-2} \end{vmatrix}$$

By applying Sylvester's identity to the determinants and exchanging some rows or columns, then

$$
\frac{
\begin{vmatrix}
-x_i^n f_i & -x^n f & -x_{i+1}^n f_{i+1} & \cdots & -x_{i+m+n-1}^n \\
1 & 1 & 1 & \cdots & 1 \\
x_i & x & x_{i+1} & & x_{i+m+n-1} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
x_i^{m-1} & x^{m-1} & x_{i+1}^{m-1} & \cdots & x_{i+m+n-1}^{m-1} \\
x_i f_i & x f & x_{i+1} f_{i+1} & \cdots & x_{i+m+n-1} f_{i+m+n-1} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
x_i^{n-1} f_i & x^{n-1} f & x_{i+1}^{n-1} f_{i+1} & \cdots & x_{i+m+n-1}^{n-1} f_{i+m+n-1} \\
0 & 1 & 0 & \cdots & 0
\end{vmatrix}
\begin{vmatrix}
1 & 1 & \cdots & 1 \\
x & x_i & \cdots & x_{i+m+n-2}^{m-1} \\
\cdots & \cdots & \cdots & \cdots \\
x^{m-1} & x_i^{m-1} & \cdots & x_{i+m+n-2}^{m-1} \\
xf & x_i f_i & \cdots & x_{i+m+n-2} f_{i+m+n-2} \\
\cdots & \cdots & \cdots & \cdots \\
x^{n-1} f & x_i^{n-1} f_i & \cdots & x_{i+m+n-2}^{n-1} f_{i+m+n-2}
\end{vmatrix}
D_{m-1,n-1}^{i+1}
}{
D_{m-1,n}^{i+1} \quad D_{m-1,n}^{i} \quad
\begin{vmatrix}
1 & 1 & \cdots & 1 \\
x & x_i & \cdots & x_{i+m+n-2}^{m-1} \\
\cdots & \cdots & \cdots & \cdots \\
x^{m-1} & x_i^{m-1} & \cdots & x_{i+m+n-2}^{m-1} \\
x f & x_i f_i & \cdots & x_{i+m+n-2} f_{i+m+n-2} \\
\cdots & \cdots & \cdots & \cdots \\
x^{n-1} f & x_i^{n-1} f_i & \cdots & x_{i+m+n-2}^{n-1} f_{i+m+n-2}
\end{vmatrix}
}
$$

After eliminating terms this is easily seen to be a constant term.

Using (2.3d,e) and the same method it is straightforward to extend the result to the cases $m<0$ or $n<0$.

## 5. NUMERICAL RESULTS

It is obvious that when $x = 0$, this algorithm reduces to the general rational extrapolation. Since the interpolating functions which are generated by this algorithm have the implicit form (refer to the Table below), it may be inconvenient to compute the interpolating function value at a given point, especially when done by the computer. But if this point $x = a$ is transformed to $x' = 0$ and the interpolation algorithm is used, the arithmetic is substantially simplified.

Example 1.

The same example is used as in [4].

Table 5.1

Rational interpolating function in an implicit form

| $i$ | $x_i$ | $R^i_{0,0}(x)$ | $R^i_{0,1}(x)$ | $R^i_{0,2}(x)$ |
|---|---|---|---|---|
| 0 | 0 | 2 | $f = -\frac{1}{3}xf+2$ | $f = \frac{1}{12}xf - \frac{5}{12}x^2f+2$ |
| 1 | 1 | $\frac{3}{2}$ | $f = -7xf+12$ | $f = -\frac{4}{3}xf - \frac{1}{3}x^2f+4$ |
| 2 | 2 | $\frac{4}{5}$ | $f = 3xf-4$ | $f = \infty$ |
| 3 | 3 | $\frac{1}{2}$ | $f = \frac{5}{3}xf-2$ | $f = \frac{28}{9}xf + \frac{1}{9}x^2f - \frac{14}{3}$ |
| 4 | 4 | $\frac{6}{17}$ | $f = \frac{37}{29}xf - \frac{714}{493}$ | |
| 5 | 5 | $\frac{7}{26}$ | | |

| $R^i_{1,0}(x)$ | $R^i_{1,1}(x)$ | $R^i_{1,2}(x)$ |
|---|---|---|
| $f = 2 - \frac{1}{2}x$ | $f = \frac{1}{7}xf+2-\frac{5}{7}x$ | $f = -x^2f+2+x$ |
| $f = \frac{11}{5} - \frac{7}{10}x$ | $f = -2xf+5-\frac{1}{2}x$ | $f = -x^2f+2+x$ |
| $f = \frac{7}{5} - \frac{3}{10}x$ | $f = 13xf-22+x$ | $f = -x^2f+2+x$. |
| $f = \frac{16}{17} - \frac{5}{34}x$ | $f = \frac{14}{5}xf-4+\frac{1}{10}x$ | |
| $f = \frac{152}{221} - \frac{37}{442}x$ | | |

| (Continued.) | | |
|---|---|---|
| $R^i_{2,0}(x)$ | $R^i_{2,1}(x)$ | $R^i_{2,2}(x)$ |
| $f = 2 - \frac{2}{5}x - \frac{1}{10}x^2$ | $f = \frac{1}{2}xf+2 - \frac{3}{2}x + \frac{x^2}{4}$ | $f = -x^2 f+2+x$ |
| $f = \frac{13}{5} - \frac{13}{10}x + \frac{1}{5}x^2$ | $f = -\frac{7}{6}xf+4 - \frac{5}{6}x + \frac{x^2}{12}$ | $f = -x^2 f+2+x$ |
| $f = \frac{158}{85} - \frac{58}{85}x + \frac{13}{170}x^2$ | $f = -\frac{33}{4}xf+17 - \frac{7}{4}x + \frac{x^2}{8}$ | |
| $f = \frac{292}{221} - \frac{163}{442}x + \frac{7}{221}x^2$ | | |
| where f is the rational approximation | | |

These results are exactly the same as those obtained explicitly in [ 4] .

The terms $Hg^i_{m,n}(x)$ and $Vg^i_{m,n}(x)$ for the construction or the interpolating function in the above table can be shown in Table 5.2.

table 5.2

| $i$ | $x_i$ | $f_i$ | $Hg_{0,1}(x)$ | | $Hg_{0,2}(x)$ | |
|---|---|---|---|---|---|---|
| 0 | 0 | 2 | $-xf$ | | $xf-x^2f$ | |
| 1 | 1 | $\frac{3}{2}$ | $-xf+\frac{3}{2}$ | | $17xf-x^2f-24$ | |
| 2 | 2 | $\frac{4}{5}$ | $-xf+\frac{8}{5}$ | | $-13xf-x^2f+24$ | |
| 3 | 3 | $\frac{1}{2}$ | $-xf+\frac{3}{2}$ | | $-13xf-x^2f+24$ | |
| 4 | 4 | $\frac{6}{17}$ | $-xf+\frac{24}{17}$ | | $-\frac{479}{29}xf-x^2f+\frac{14280}{493}$ | |
| 5 | 5 | $\frac{7}{26}$ | $-xf+\frac{35}{26}$ | | | |

| $Vg_{1,0}(x)$ | $Hg_{1,1}(x)$ | $Vg_{1,1}(x)$ | $Hg_{1,2}(x)$ | $Vg_{1,2}(x)$ |
|---|---|---|---|---|
| $-x$ | $-xf+\frac{3}{2}x$ | $\frac{2}{3}xf-x$ | $-\frac{1}{7}xf-x^2f+\frac{12}{7}$ | $\frac{1}{12}xf+\frac{7}{12}x^2f-x$ |
| $1-x$ | $-xf+\frac{7}{5}+\frac{1}{10}x$ | $10xf-14-x$ | $2xf-x^2f-3+\frac{3}{2}x$ | $-\frac{4}{3}xf+\frac{2}{3}x^2f+2-x$ |
| $2-x$ | $-xf+\frac{9}{5}-\frac{1}{10}x$ | $-10xf+18-x$ | $-13xf-x^2f+24$ | $\infty$ |
| $3-x$ | $-xf+\frac{30}{17}-\frac{3}{34}x$ | $-\frac{34}{3}xf+20-x$ | $-\frac{14}{5}xf-x^2f+6-\frac{153}{170}x$ | $-\frac{476}{153}xf+\frac{170}{153}x^2f-\frac{340}{51}-x$ |
| $4-x$ | $-xf+\frac{370}{221}-\frac{29}{442}x$ | $-\frac{442}{29}xf+\frac{740}{29}-x$ | | |
| $5-x$ | | | | |

| $Vg_{2,0}(x)$ | $Hg_{2,1}(x)$ | $Vg_{2,1}(x)$ | $Hg_{2,2}(x)$ | $Vg_{2,2}(x)$ |
|---|---|---|---|---|
| $x-x^2$ | $-xf+\frac{11}{5}x-\frac{7}{10}x^2$ | $-\frac{10}{7}xf+\frac{22}{7}x-x^2$ | $-\frac{1}{2}xf-x^2f+\frac{5}{2}x-\frac{1}{4}x^2$ | $-2xf-4x^2f+10x-x^2$ |
| $-2+3x-x^2$ | $-xf+\frac{6}{5}+\frac{2}{5}x-\frac{1}{10}x^2$ | $-10xf+12+4x-x^2$ | $\frac{7}{6}xf-x^2f-2+\frac{11}{6}x-\frac{1}{12}x^2$ | $14xf-12x^2f-24+22x-x^2$ |
| $-6+5x-x^2$ | $-xf+\frac{156}{85}-\frac{11}{85}x+\frac{1}{170}x^2$ | $170xf-312+22x-x^2$ | $\frac{33}{4}xf-x^2f-15+\frac{11}{4}x-\frac{1}{8}x^2$ | $66xf-8x^2f-120+22x-x^2$ |
| $-12+7x-x^2$ | $-xf+\frac{420}{221}-\frac{37}{221}x+\frac{5}{442}x^2$ | $\frac{442}{5}xf-168+\frac{74}{5}x-x^2$ | | |
| $-20+9x-x^2$ | | | | |

Numerical rational interpolation at x = 3.5, computed by transforming x = 3.5 to x' = 0.

Table 5.3

| $x'_i$ | $R_{0,0}$ | $R_{0,1}$ | $R_{0,2}$ |
|---|---|---|---|
| -3.5 | 2 | 0.92307692 | 0.34408602 |
| -2.5 | $\frac{3}{2}$ | 0.47058824 | 0.45714286 |
| -1.5 | $\frac{4}{5}$ | 0.42105263 | $\infty$ |
| -0.5 | $\frac{1}{2}$ | 0.41379310 | 0.41481481 |
| 0.5 | $\frac{6}{17}$ | 0.41791045 | |
| 1.5 | $\frac{7}{26}$ | | |

| $R_{1,0}$ | $R_{1,1}$ | $R_{1,2}$ |
|---|---|---|
| 0.25 | -1.0 | 0.41509434 |
| -0.25 | 0.40625 | 0.41509434 |
| 0.35 | 0.41573034 | 0.41509434 |
| 0.42647059 | 0.41477273 | |
| 0.39479638 | | |

| $R_{2,0}$ | $R_{2,1}$ | $R_{2,2}$ |
|---|---|---|
| -0.625 | 0.25 | 0.41509434 |
| 0.5 | 0.41393443 | 0.41509434 |
| 0.40735294 | 0.41527197 | |
| 0.41855204 | | |

Example 2.

Since $Hg^{2}_{0,1}(x) = Hg^{3}_{0,1}(x)$, singularity occurs at $Vg^{2}_{1,1}(x)$ and $Hg^{2}_{0,2}(x)$. The approximation $R^{2}_{0,1}(x)$ does not exist, but by using the rules of Section 3 the algorithm can continue to calculate higher order rational approximations.

Table 5.4

| $i$ | $x_i$ | $f_i$ | $Hg_{0,1}(x)$ | | $Hg_{0,2}(x)$ | |
|---|---|---|---|---|---|---|
| 0 | 0.5 | 2 | $-xf+1$ | | $\frac{23}{10}xf-x^2f-\frac{9}{5}$ | |
| 1 | 2 | 3 | $-xf+6$ | | $xf-x^2f+6$ | |
| 2 | 4 | 0.5 | $-xf+2$ | | $\infty$ | |
| 3 | 1 | 2 | $-xf+2$ | | $4xf-x^2f+6$ | |
| 4 | 3 | 2 | $-xf+6$ | | | |
| | $Vg_{1,0}(x)$ | | $Hg_{1,1}(x)$ | $Vg_{1,1}(x)$ | $Hg_{1,2}(x)$ | $Vg_{1,2}(x)$ |
| | $0.5-x$ | | $-xf-\frac{2}{3}+\frac{10}{3}x$ | $\frac{3}{10}xf+0.2-x$ | $\frac{87}{48}xf-x^2f-\frac{51}{24}+\frac{39}{24}x$ | $\frac{87}{78}xf-\frac{24}{39}x^2f+\frac{51}{39}-x$ |
| | $2-x$ | | $-xf+10-2x$ | $-\frac{1}{2}xf+5-x$ | $2xf-x^2f-4+2x$ | $xf-0.5x^2f-2+x$ |
| | $4-x$ | | $-xf+2$ | $\infty$ | $3xf-x^2f-6+2x$ | $\frac{3}{2}xf-\frac{1}{2}x^2f-3+x$ |
| | $1-x$ | | $-xf+2x$ | $\frac{1}{2}xf-x$ | | |
| | $3-x$ | | | | | |

$R^i_{m,n}(x)$ can be computed either from $R^i_{m-1,n}(x)$ or $R^i_{m,n-1}(x)$

Table 5.5

| $i$ | $x_i$ | $R_{0,0}(x)$ | $R_{0,1}(x)$ | $R_{0,2}(x)$ |
|---|---|---|---|---|
| 0 | 0.5 | 2 | $f = \frac{1}{5}xf + \frac{9}{5}$ | $f = \frac{99}{104}xf - \frac{17}{52}x^2f + \frac{63}{52}$ |
| 1 | 2 | 3 | $f = \frac{5}{8}xf - \frac{6}{8}$ | $f = \frac{7}{8}xf - \frac{1}{4}x^2f + \frac{3}{4}$ |
| 2 | 4 | 0.5 | $\infty$ | $f = xf - \frac{1}{4}x^2f + \frac{1}{2}$ |
| 3 | 1 | 2 | | |
| 4 | 3 | 2 | $f = 2$ | |

| $R_{1,0}(x)$ | $R_{1,1}(x)$ | $R_{1,2}(x)$ |
|---|---|---|
| $f = \frac{5}{3} + \frac{2}{3}x$ | $f = \frac{23}{64}xf + \frac{61}{32} - \frac{51}{96}x$ | $f = \frac{5}{24}xf + \frac{1}{12}x^2f + \frac{25}{12} - \frac{2}{3}x$ |
| $f = \frac{11}{2} - \frac{5}{4}x$ | $f = \frac{3}{8}xf + \frac{7}{4} - \frac{1}{2}x$ | $f = \frac{5}{8}xf - \frac{1}{8}x^2f + \frac{5}{4} - \frac{1}{4}x$ |
| $f = \frac{5}{2} - \frac{1}{2}x$ | $f = \frac{1}{4}xf + 2 - \frac{1}{2}x$ | |
| $f = 2$ | | |

Example 3.

In this example $f(x_2) = 0$ and hence $R^i_{0,1}(x)$, $i = 1,2$ and $R^i_{0,2}(x)$, $i = 0,1,2$ do not have an explicit form. In addition $Hg^0_{11}(x) = Hg^1_{1,1}(x)$ and $Vg^0_{1,1}(x) = Vg^1_{1,1}(x)$ which lead to singularities at $Vg^0_{2,1}(x)$ and $Hg^0_{1,2}(x)$. Hence $R^0_{1,1}(x)$ does not exist. Note that the algorithm continues to calculate higher order interpolations in both cases.

Table 5.6

| $i$ | $x_i$ | $f_i$ | $Hg_{0,1}(x)$ | | $Hg_{0,2}(x)$ | |
|---|---|---|---|---|---|---|
| 0 | 1 | 4 | $-xf+4$ | | $-x^2f+4$ | |
| 1 | 2 | 1 | $-xf+2$ | | $2xf-x^2f$ | |
| 2 | 3 | 0 | $-xf$ | | $4xf-x^2f$ | |
| 3 | 4 | 1 | $-xf+4$ | | $-\frac{17}{3}xf+x^2f-\frac{20}{3}$ | |
| 4 | 5 | 2 | $-xf+10$ | | | |

| $Vg_{1,0}(x)$ | $Hg_{1,1}(x)$ | $Vg_{1,1}(x)$ | $Hg_{1,2}(x)$ | $Vg_{1,2}(x)$ |
|---|---|---|---|---|
| $1-x$ | $-xf+6-2x$ | $-\frac{1}{2}xf+3-x$ | $\infty$ | $-\frac{1}{2}xf+3-x$ |
| $2-x$ | $-xf+6-2x$ | $-\frac{1}{2}xf+3-x$ | $\frac{10}{3}xf-x^2f-8+\frac{8}{3}x$ | $-\frac{5}{4}xf+\frac{3}{8}x^2f+3-x$ |
| $3-x$ | $-xf-12+4x$ | $\frac{1}{4}xf+3-x$ | $9xf-x^2f+60-2x$ | $\frac{9}{20}xf-\frac{1}{20}x^2f+3-x$ |
| $4-x$ | $-xf-20+6x$ | $\frac{1}{6}xf+\frac{10}{3}-x$ | | |
| $5-x$ | | | | |

| $Vg_{2,0}(x)$ | $Hg_{2,1}(x)$ | $Vg_{2,1}(x)$ | $Hg_{2,2}(x)$ | $Vg_{2,2}(x)$ |
|---|---|---|---|---|
| $-2+3x-x^2$ | $-xf+6-2x$ | $\infty$ | $4xf-x^2f-24+14x-2x^2$ | $2xf-\frac{1}{2}x^2f-12+7x-x^2$ |
| $-6+5x-x^2$ | $-xf+24-17x+3x^2$ | $\frac{1}{3}xf-8+\frac{17}{3}-x^2$ | $\frac{1}{2}xf-x^2f+60-\frac{91}{2}x+\frac{17}{2}x^2$ | $-\frac{1}{17}xf+\frac{2}{17}x^2f-\frac{120}{17}+\frac{91}{17}x-x^2$ |
| $-12+7x-x^2$ | $-xf-3x+x^3$ | $xf+3x-x^2$ | | |
| $-20+9x-x^2$ | | | | |

Table 5.7

| $i$ | $x_i$ | $R_{0,0}(x)$ | $R_{0,1}(x)$ | $R_{0,2}(x)$ |
|---|---|---|---|---|
| 0 | 1 | 4 | $f = -2+\frac{3}{2}xf$ | $f = \frac{3}{2}xf-\frac{1}{2}x^2f$ |
| 1 | 2 | 1 | $f = \frac{1}{2}xf$ | $f = \frac{3}{4}xf-\frac{1}{8}x^2f$ |
| 2 | 3 | 0 | $f = \frac{1}{4}xf$ | $f = \frac{9}{20}xf-\frac{1}{20}x^2f$ |
| 3 | 4 | 1 | $f = \frac{1}{3}+\frac{1}{6}xf$ | |
| 4 | 5 | 2 | | |
| | | $R_{1,0}(x)$ | $R_{1,1}(x)$ | $R_{1,2}(x)$ |
| | | $f = 7-3x$ | $\infty$ | $f = 2xf-\frac{1}{2}x^2f-3+x$ |
| | | $f = 3-x$ | $f = \frac{1}{3}xf+1-\frac{1}{3}x$ | $f = \frac{9}{17}xf-\frac{1}{17}x^2f+\frac{9}{17}-\frac{3}{17}x$ |
| | | $f = -3+x$ | $f = -3+x$ | |
| | | $f = -3+x$ | | |
| | | $R_{2,0}(x)$ | $R_{2,1}(x)$ | $R_{2,2}(x)$ |
| | | $f = 9-6x+x^2$ | $f = 9-6x+x^2$ | $f = \frac{4}{7}xf-\frac{1}{7}x^2f+\frac{39}{7}-4x+\frac{5}{7}x^2$ |
| | | $f = 9-6x+x^2$ | $f = \frac{1}{2}xf-3+\frac{5}{2}x-\frac{1}{2}x^2$ | |
| | | $f = -3+x$ | | |

Example 4.

If $x_i$ is considered for $i = 0,1,2,3$ in Example 1, the interpolating functions may be expressed explicitly. Then $R^0_{1,0}(x) = \frac{4-x}{2}$; $R^0_{1,1}(x) = \frac{14-5x}{7-x}$, $R^0_{2,1}(x) = \frac{4-x}{2}$ for the sequence of functions which can be expressed as continued fractions. Note that $x_2 = 2$ is considered as an 'unattainable point' of $R^0_{2,1}(x)$ (see [10]).

If the method suggested in [3] which reorders the points so that the 'unattainable points' appear at the end of the list is used, then

$$R^0_{1,0}(x) = \frac{4-x}{2} \quad \text{and} \quad R^0_{1,1}(x) = \frac{14-5x}{7-x} .$$

In this case $x_3 = 3$ is considered an unattainable point and the algorithm terminates.

With the algorithm described in this chapter, all possible forms of the interpolating function may be computed and decided which is most suitable. Even when more points are added, the interpolation procedure may still be continued without reordering the points and higher order functions may be obtained. For example if $f(4) = \frac{6}{17}$ is added, then $R^0_{2,2}(x) = \frac{2+x}{1+x^2}$ ( see Example 1) Alternatively suppose $f(4) = 1$ is added, then $R^0_{2,2}(x) \frac{96-62x+11x^2}{48-22x+4x^2}$. Both of these functions interpolate $f(x)$ at all the points $x_i$ $i = 0,1,2,3,4$ despite the existence of unattainable points for lower order interpolations.

Example 5. (Example 3 in [3]).

If f is infinite at one of the interpolation points, It can be replaced by a parameter $\alpha$, in the computation.

Suppose $f(0) = 1$, $f(1) = 0$, $f(2) = \infty$. Then

Table 5.8

| $i$ | $x_i$ | $f_i$ | $Hg_{0,1}(x)$ |
|---|---|---|---|
| 0 | 0 | 1 | $-xf$ |
| 1 | 1 | 0 | $-xf$ |
| 2 | 2 | $\alpha$ | $-xf+2\alpha$ |

| $Vg_{1,0}(x)$ | $Hg_{1,1}(x)$ |
|---|---|
| $-x$ | $-xf$ |
| $1-x$ | $-xf-2\alpha(1-x)$ |
| $2-x$ | |

The rational interpolants are

| $i$ | $R_{1,0}(x)$ | $R_{1,1}(x)$ |
|---|---|---|
| 0 | $f = 1-x$ | $f = \frac{1+\alpha}{2\alpha}xf+1-x$ |
| 1 | $f = -\alpha(1-x)$ | |

Hence $R^0_{1,1}(x) = \dfrac{1-x}{1-\frac{1}{2}x-\frac{1}{2\alpha}x}$

Setting $\frac{1}{\alpha} = 0$, It is noted that $R^0_{1,1}(x) = \dfrac{1-x}{1-\frac{1}{2}x}$ correctly interpolates the data.

## 6 CONCLUSION

An algorithm for the recursive calculation of the interpolating rational function has been given. The algorithm has been constructed in a general form to allow for the expression of the rational function in any suitable basis subject to the Chebyshev condition being satisfied. The algorithm is given a form which lends itself to generalization.

The way of constructing the coefficients of the interpolating function by this algorithm is quite easy, e.g. $R^i_{m,n}(x)$ is obtained either from $R^i_{m-1,n}(x)$ by adding a multiple of $Vg^i_{m,n}(x)$ or from $R^i_{m,n-1}(x)$ by adding a multiple of $Hg^i_{m,n}(x)$. These multiples in fact are the constant ratios of the differences of the adjacent terms of $Vg^i_{m,n}(x)$ or $Hg^i_{m,n}(x)$ even in the singular cases.

It has been shown that an isolated singularity may be avoided by 'jumping over' those situations where a rational interpolating function does not exist without the necessity of restarting the calculation for a different ordering of the interpolating points.

The evaluation of the rational interpolating approximation at a particular point $x = a$ is more efficiently accomplished by a transformation.

The general extrapolation algorithm has been derived independently in [4],[5],[8],[9]. The MNA algorithm is quite similar to the algorithm in [8]. The algorithm for rational extrapolation in [8] computes the numerator and denominator separately. The algorithm in this paper computes the rational extrapolation in an implicit form.

REFERENCES


1. C. BREZINSKI, A general extrapolation Algorithm, *Numer. Math.* **35**, (1980), 175-187.

2. C. BREZINSKI, The Muhlbach-Neville-Aitken Algorithm and some some extensions, *B.I.T.* **20**, (1980), 444-451.

3. P. R. GRAVES-MORRIS and T. R. HOPKINS, Reliable Rational Interpolation, *Numer. Math.* **36**, (1981), 111-128.

4. T. HAVIE, Generalized Neville type extrapolation schemes. *B.I.T.* **19**, (1979), 204-213.

5. T. HAVIE, Remarks on a unified theory for classical and generalized interpolation and extrapolation, *B.I.T.* **21**, (1981) 465-474.

6. F. M. LARKIN, Some techniques for rational interpolation, *Computer J.* **10**, (1967), 178-187.

7. G. MUHLBACH, The general Neville-Aitken Algorithm and some Applications, *Numer. Math.* **31**, (1978) 97-110.

8. C. SCHNEIDER, Vereinfachte Rekursionen zur Richardson-Extrapolation in Spezialfällen, *Numer. Math.*, **24**, (1975), 177-184.

9. J. WIMP, Sequence transformations and their applications,
*Academic Press*, N.Y. 1981.

10. LUC WUYTACK, On some aspects of the rational interpolation
Problem,
*Siam J. Num. Anal.* **11**, (1974), 52-60.

11. P. WYNN, Singular rules for certain non-linear Algorithm,
*B.I.T.* **3**, (1963), 175-195.